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Fourier transform construction by vector graphics

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The learning process is invariably improved by alternative, complementary views of a concept. This paper's pictorial representation of the discrete Fourier transform (DFT) is helpful in that students can understand internal steps of development, as well as the final result, in terms of vectors. It builds upon the widely known fact that the exponential terms in the transform pair are roots of unity in the complex plane; or in alternative physics terminology, they are unit vectors. After weighting these vectors by sampled values of the function to be transformed, using a simple recipe, the DFT is obtained through vector addition.

I. INTRODUCTION

The Fourier transform has received greater attention in experimental physics as instruments based on it have become popular.[1-3] Meyer-Arendt[4] displays boldness in his statement: "Fourier transform spectroscopy is the superior method. Even more important, Fourier spectroscopy is not simply the application of another little invention; rather, it marks a turning point in philosophy, away from high-precision delicate optics, toward a simple, rugged sensor coupled with sophisticated electronic data processing." Many disciplines other than optics have also been assisted by this powerful mathematical tool, since it is now possible to perform rapid conversion of time traces to the frequency domain, using the fast Fourier transform (FFT).[5] For example, power spectra have become central to the study of systems displaying deterministic chaos.[6,7] In the realm of image analysis, some engineers have focused much of their career on computer techniques based on the two-dimensional discrete Fourier transform (DFT).[8] In all of these examples, which represent a very small fraction of the whole world of Fourier processing, the advent of inexpensive digital computers was prerequisite to making the necessary computations practical. Numerous algorithms based on the FFT are now available to take an analog to digital converted voltage versus time record and produce a spectrum from it. Typically, these records are at least 1024 samples, if the resolution is to be reasonable.

The present paper describes an unconventional way of viewing the DFT. It facilitates understanding for those who are best served by visual aids. It has been successfully used by the author for the past 4 years in teaching an undergraduate optics course, as well as an experimental laboratory in computational physics. As opposed to the "abstraction" of the complex exponential representation, it is based on vectors. Most professionals with whom the matter has been discussed have recognized that the resultant produced by any DFT algorithm can be thought of as a set of vectors. The author is not aware, however, of anyone else having used vectors in this way for its development.

II. THEORY

For the present paper, the Fourier transform pair in x and k is defined as

$$G(k) = \int_{-\infty}^{\infty} g(x)e^{-ikx}\,dx, \tag{1a}$$

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(k)e^{ikx}\,dk. \tag{1b}$$

As noted in Guenther,[9] there are alternative forms in which (1) the pair of equations is symmetric, by associating $(2\pi)^{-1/2}$ with each one; and/or (2) the positive and negative exponentials are interchanged. The six different forms (all acceptable) have been the source of confusion for many students through the years.

The variable k, which is conjugate to the position variable x, involves the spatial frequency f_x as follows:

$$k = 2\pi f_x = 2\pi/X, \tag{2}$$

where X is the length of the interval being transformed. Students typically encounter the Fourier transform pair for the first time in terms of time t and angular frequency $\omega = 2\pi f$. Though irrelevant to mathematicians, the present notation is purposely directed toward optical applications.

For numerical evaluation, Eq. (1) is usually made discrete in terms of f_x rather than k, as follows:

$$G\left(\frac{J}{X}\right) = \frac{X}{N}\sum_{m=0}^{N-1} g\left(\frac{mX}{N}\right)e^{-i2\pi Jm/N},$$

$$J = 0,1,2,\ldots,N-1, \tag{3a}$$

$$g\left(\frac{LX}{N}\right) = \frac{1}{X}\sum_{n=0}^{N-1} G\left(\frac{n}{X}\right)e^{i2\pi Ln/N}, \quad L = 0,1,2,\ldots,N-1, \tag{3b}$$

where X is the total distance over which the transform is approximated, and the space domain function $g(x)$ is sampled at N equispaced points. The frequency when $J = N/2$ is the Nyquist folding frequency,[10] and for both the J and L integers, values greater than $N/2$ correspond to negative values for the associated variables.

It is convenient, and consistent with the usual radix 2 FFT, to choose the number of sample points according to the relation $N = 2^\gamma$, where γ is an integer. This is illustrated in Fig. 1 for the case of $N = 8$. The eight roots of unity for this case are the points of the arrowheads. The vector (phasor) convention has been purposely emphasized, because this set of vectors will be used in later examples of transform construction by vector graphics. The examples are primarily of pedagogical value, since practical algorithms typically use 1024 or more points. The increase of resolution as N increases would result in prohibitively long execution times were it not for the FFT.

For purposes of simplification in all of the examples which follow, the scaling will be selected such that $X/N = 1$. This is equivalent to choosing a distance unit such that the length of the interval which is to be transformed is just equal to the number of samples.

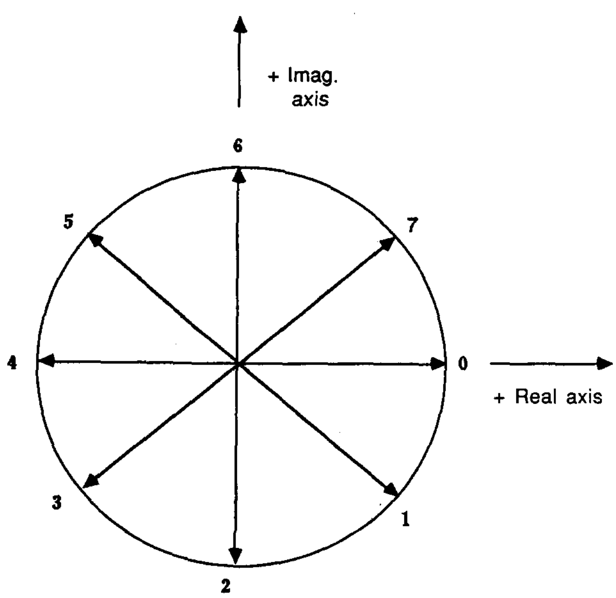

Fig. 1. Roots of unity in the complex plane in terms of unit vectors; the case shown is for $N = 8$.

Further let $N = 8$, and consider the $J = 0$, or dc component, of Eq. (3a):

$$G(0) = [g(0) + g(1) + \cdots + g(7)]e^{-i0}. \tag{4a}$$

The expression e^{-i0} corresponds to the unit vector labeled **0** in Fig. 1 (with the convention of representing unit vectors by boldface type). Perhaps the simplest way to make this association is to use Euler's identity $e^{i\phi} = \cos\phi + i\sin\phi$. We thus recognize the dc component to be the unit vector **0** multiplied by the sum of the eight parts of $g(i)$.

Now consider the $J = 1$, or fundamental component:

$$G(1/8) = g(0)e^{-i2\pi 0/8} + g(1)e^{-i2\pi 1/8} + g(2)e^{-i2\pi 2/8} + \cdots + g(7)e^{-i2\pi 7/8}. \tag{4b}$$

The exponentials are just the eight unit vectors of Fig. 1, in sequence, so that the expression can be rewritten,

$$G(1/8) = g(0)\mathbf{0} + g(1)\mathbf{1} + g(2)\mathbf{2} + \cdots + g(7)\mathbf{7}. \tag{4b'}$$

Thus the fundamental is obtained through the addition of a vector set, the eight parts of which are generated by weighting the eight unit vectors by the sequential values of g.

In a similar manner, it can be readily shown that the $J = 2$, or second harmonic, is

$$G(2/8) = g(0)\mathbf{0} + g(1)\mathbf{2} + g(2)\mathbf{4} + g(3)\mathbf{6} + g(4)\mathbf{0} + g(5)\mathbf{2} + g(6)\mathbf{4} + g(7)\mathbf{6}, \tag{4c}$$

where the roots of unity circle has been traversed twice, and the eight unit vectors are weighted, every second one, by the eight values of g. Likewise, the third harmonic is obtained by going around the circle three times, and weighting every third vector by the sequential values of g.

Extending this process to N samples and thus N unit vectors

$$G\left(\frac{j}{N}\right) = \sum_{i=0}^{N-1} g(i)\mathbf{j{\cdot}i}, \tag{5}$$

where $\mathbf{j{\cdot}i} = \mathrm{mod}(\mathbf{j{\cdot}i},N)$ is implied for all unit vectors associated with multiple traversals of the circle.

III. EXAMPLE DFTs USING THE VECTOR GRAPHICS CONSTRUCTION

A. Delta functions at $+x_0$ and $-x_0$

The discrete transform gives exact results for this case, even for small N. Figure 2 details the use of Eq. (5) with $N = 8$ and the "delta" functions at $+1$ and -1. [A true delta function cannot be sampled since it is of unit area and infinite height. Here the single point representing $\delta(x - x_0)$ is of unit height.] Because it is an even function that has been transformed, the result is real. The continuous case is shown as the cosine curve in Fig. 2(c), and the DFT points are the solid circles.

B. Exponential

For this case, $g(x) = e^{-x}$ for x positive and $g(x) = 0$ for x negative. One anticipates that the transform must have both real and imaginary parts, since $g(x)$ is neither even nor odd. The results are shown in Fig. 3. Note that $g(0)$ has been set to 0.5, i.e., the midvalue, in keeping with standard Fourier processing practice at a discontinuity. In comparing with the exact, continuous transform case, which in this case is easily obtained, one must be careful to account properly for the factor of 2π.

In contrast with the delta function case, it is seen that the

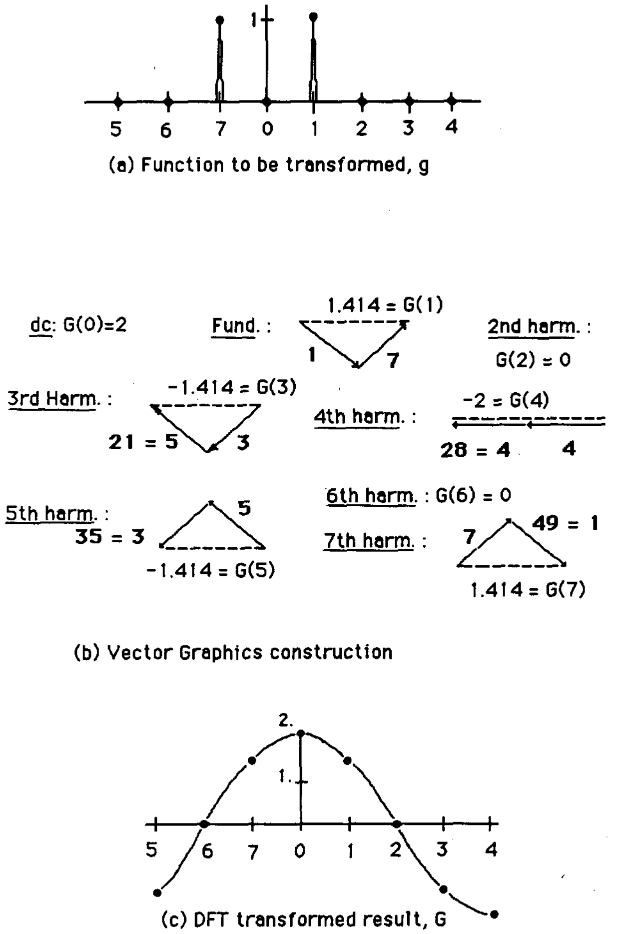

Fig. 2. Example discrete Fourier transform ($N = 8$) of symmetric delta functions.

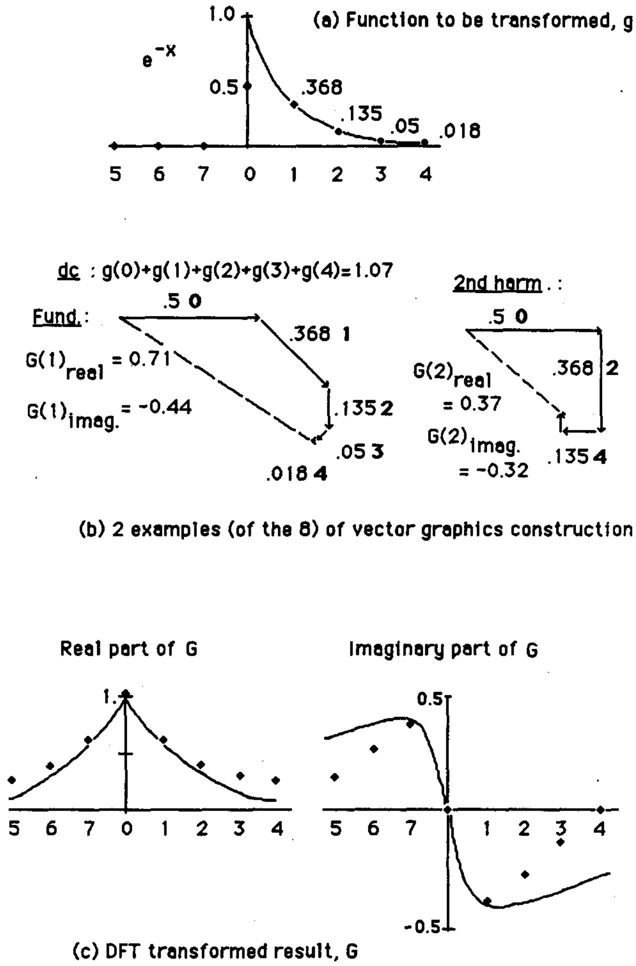

Fig. 3. Example discrete Fourier transform ($N = 8$) of an exponential.

small-sample discrete transform is not able to estimate accurately the high-frequency components. This is especially true of the imaginary part. The errors are not a failure of the graphics technique, but rather are due to the fact that the DFT is, in general, an ever poorer approximation to the continuous transform as the sample number decreases toward zero.

IV. CONCLUSION

An alternative and complementary view of the discrete Fourier transform has been presented. Based in vector graphics, it develops a spectrum of $N = 2^{\gamma}$ parts, where γ is an integer. It does this by weighting N unit vectors, which are roots of unity, and then finding their sum. The angular frequency of the "phasor" which rotates on the unit circle determines the harmonic which is to be developed. For the fundamental, as an example, the N samples of the function which is being transformed "modulates" this phasor as it undergoes a single rotation of the circle at constant angular velocity. For the second harmonic, the angular velocity is doubled, and the circle is traversed twice as the phasor is again modulated by the same N samples of the function. To obtain the complete transform, this process is repeated through $N - 1$, to give a resultant N vectors.

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