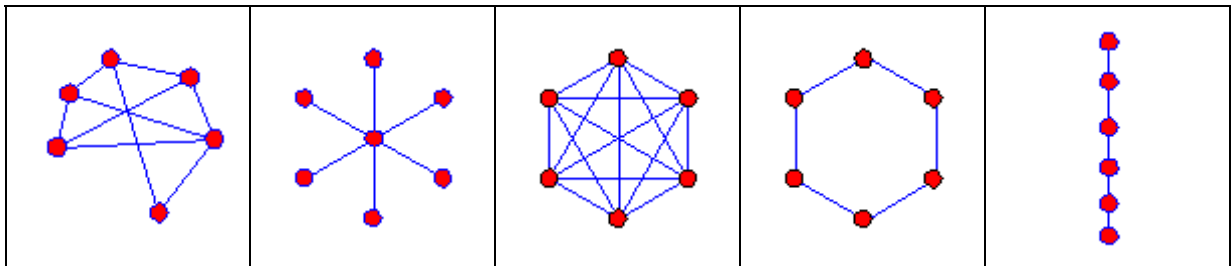# Causal Sets, Hypergraphs and Cosmology

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Causal sets are discrete structures consisting of points and causal links; they show great promise as the starting point for quantization of spacetime and general relativity. A graph is a discrete set of nodes and connecting links. A hypergraph is a generalized graph, wherein every subset of the node set may be included as an edge, which is analogous to a (2-node) link. A fundamental correspondence is presented between causal sets and hypergraphs. This is used to define a time index for each causal set point, which well-orders the set, as well as a spatial distance between points, which obeys the triangle inequality. The complete hypergraph is considered as the prime example. The time and spatial measures provide local and global structure for the corresponding causal set, as well as a simple derivation of the Hubble Law.

Motivation for this Work

Causal Sets and Hypergraphs

What is a causal set?

Physical interpretation of causal sets

What are graphs and hypergraphs?

Correspondence between causal sets and hypergraphs

Cosmology

The Omega Set

Three results for the Omega Set

Time index: global shape and central bulge

Distance measure: triangle inequality and local shape

The Hubble Law

Outlook for Future Work

What is a causal set?

A causal set is a set of points (labeled by **α, β, γ…**), together with a binary relation R with the following properties:
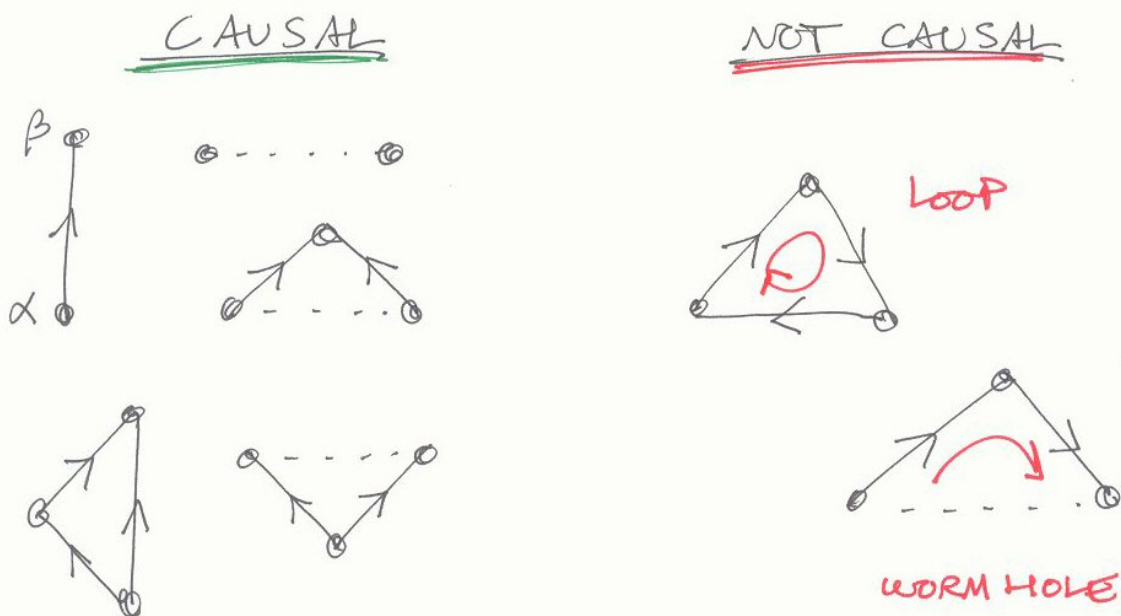
R ={ordered pairs (**αβ**)}

R is <u>anti-symmetric</u>:

    If (**αβ**) is in R, then (**βα**) is not in R.

R is <u>transitive</u>:

    If (**αβ**) is in R and (**βγ**) is in R, then (**αγ**) is in R.

R is <u>locally finite</u>:

    A point **γ** is said to lie between **α** and **β** if (**αγ**) and (**γβ**) are both in R. For every pair (**αβ**), at most a finite number of points lie between them.

Physical interpretation of causal sets

R determines the flow of information. If $(\alpha\beta)$ is in R, then α precedes β in a logical sense, so information can move from α to β. Suppose there is information (a state) associated with each point of the causal set. Then, the state at any point β depends on the states at all points preceding it, i.e. all those points α such that $(\alpha\beta)$ is in R. The state at α can affect the state at β, but the state at β cannot affect the state at α.

In terms of relativity, R provides a (partial) light cone structure. Relative to any point α, the causal set is divided into four regions:

    I)   the point α itself,
    II)   the <u>present</u>, all points β such that neither $(\alpha\beta)$ nor $(\beta\alpha)$,
    III)   the <u>future</u>, all points β such that $(\alpha\beta)$, and
    IV) the <u>past</u>, all points β b such that $(\beta\alpha)$.

Note that R cannot distinguish between the future light cone and the interior of the future light cone, nor between the past light cone and the interior of the past light cone. This may be resolved in light of the correspondence between causal sets and graphs given below.
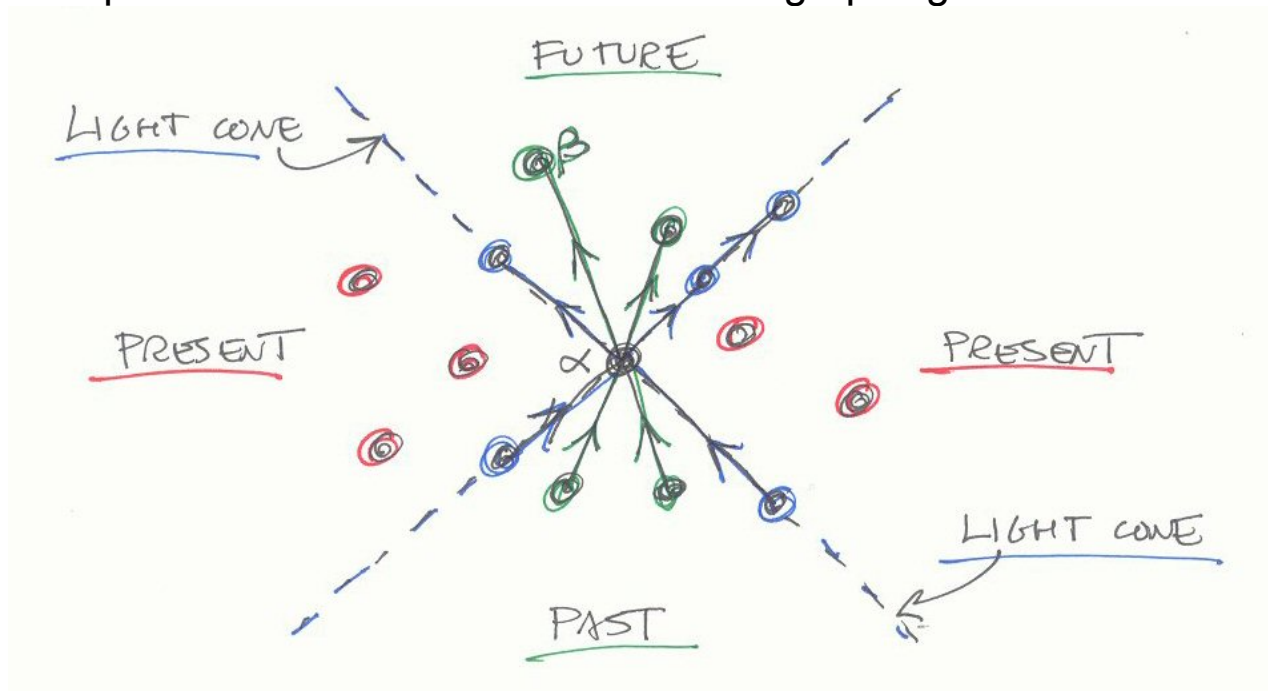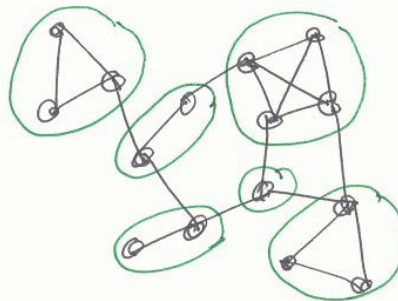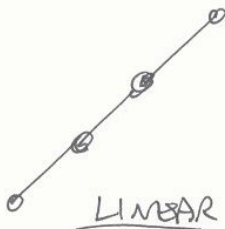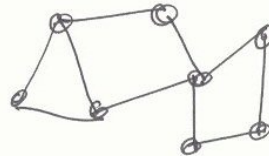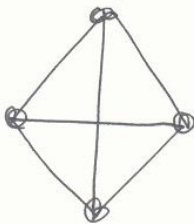
What are graphs and hypergraphs? (1)

A graph is a set of <u>nodes</u> (labeled by i, j, k … = 1 to N) together with <u>links</u> (ij) and <u>weights</u> w(ij). We consider only finite simple symmetric graphs: the number N is finite and fixed, all weights w(ij)= 0 or 1, and w(ij)=w(ji).

The flow of information among the nodes in a graph is based on the adjacency matrix, which is composed of the weights w(ij), with w(ii)=0. In a quantum mechanical context one may use the Schroedinger equation for time evolution and the Laplacian matrix for kinetic energy. We will not pursue this here. Instead we consider a generalization of the simple graph related to causal sets: the hypergraph.

Specification of a graph consists of choosing for every pair of nodes either a 0 or a 1. That is the same as picking one subset of the set of all unordered node pairs. If a node pair is in the set, it receives a weight w(ij)=1, otherwise w(ij)=0.
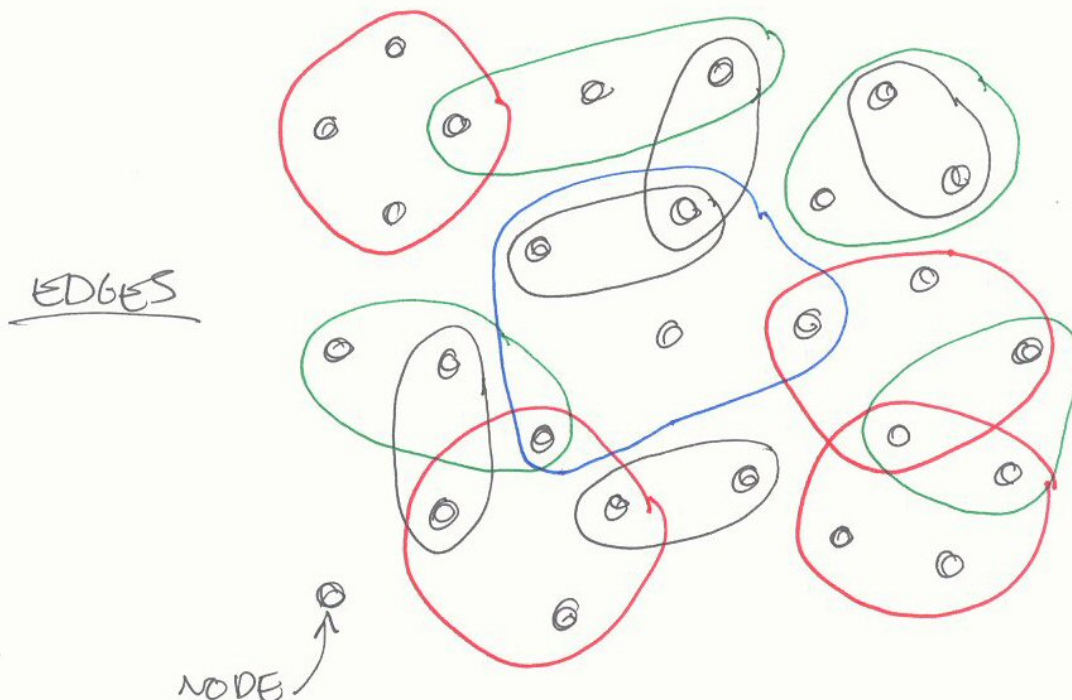
What are graphs and hypergraphs? (2)

To specify a hypergraph we choose from the set of all <u>edges</u> of order N, i.e. unordered subsets of the set of N nodes. The set of edges is also the power set of N, P(N). Thus a hypergraph is a subset of P(N). Note that P(N) includes not only pairs of nodes, but also singletons (the nodes themselves), triples and other k-tuples where k ≤ N, as well as the null set and the entire set consisting of N nodes.

Consider the size of these sets. A graph has N nodes. The number of links is N(N-1)/2. The number of edges is 2^N.   To make one graph we must choose 0 or 1 for each link, so there are $2^{N(N-1)/2}$ possible graphs. To make a hypergraph we must choose 0 or 1 for each edge, so there are 2 to the power 2^N possible hypergraphs. This is large for even modest values of N; a realistic value of N will turn out to be about 10^{60}…

Correspondence between causal sets and hypergraphs

Corresponding to the relation R between causal set points is the relation of set inclusion between edges in the hypergraph. For any two edges A and B, let the corresponding points in the causal set be **α** and **β**. We say that A ⊆ B if A is a subset of B. Now if A ⊆ B then (**αβ**) is in R, i.e. **α** precedes **β**.

This relation is anti-symmetric, locally finite (if N is finite) and transitive by the usual properties of set inclusion.

<u>For every hypergraph there is a unique corresponding causal set.</u> For every causal set there is a class of hypergraphs that can represent it and a smallest value of N that will suffice. The hypergraph is not unique because any given node can always be accompanied by additional "companion" nodes; these are included in every edge that includes the original node. So a hypergraph is a kind of "microstate" and the causal set is a "macrostate."

If neither A ⊆ B nor B ⊆ A, there are two possibilities: A and B may have no nodes in common, or they may partially overlap. Either way (**αβ**) is not in R. Thus the hypergraphs provide an additional structure beyond that specified by the causal set itself.
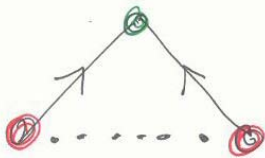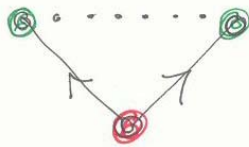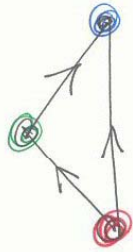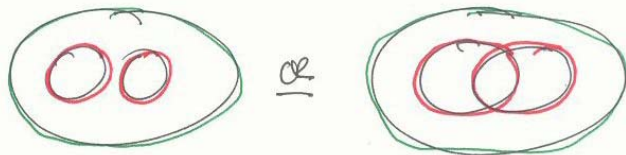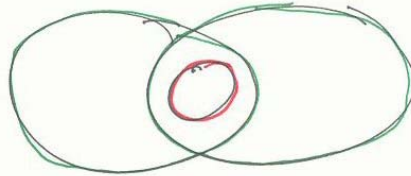
CAUSAL SET   HYPERGRAPH
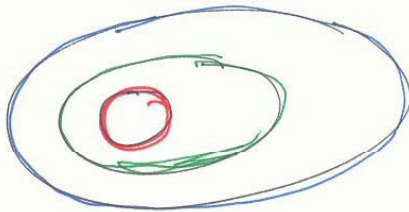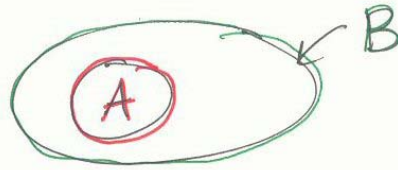
The Omega Set

As an example of a causal set cosmology based on hypergraphs, we will use the simplest non-trivial case: the Omega Set. This is the causal set corresponding to the hypergraph containing <u>every possible edge over N nodes</u>. Any causal set that corresponds to a hypergraph over N nodes is a subset of the Omega Set. Thus it is a kind of "envelope" for causal sets of order N.

$$\Omega(3)$$

CAUSAL SET

$$k = \begin{array}{c} 3 \\ 2 \\ 1 \end{array}$$

HYPERGRAPH

$$+ \phi = \{ \; \}$$

(SOME CAUSAL LINKS DELETED FOR CLARITY)
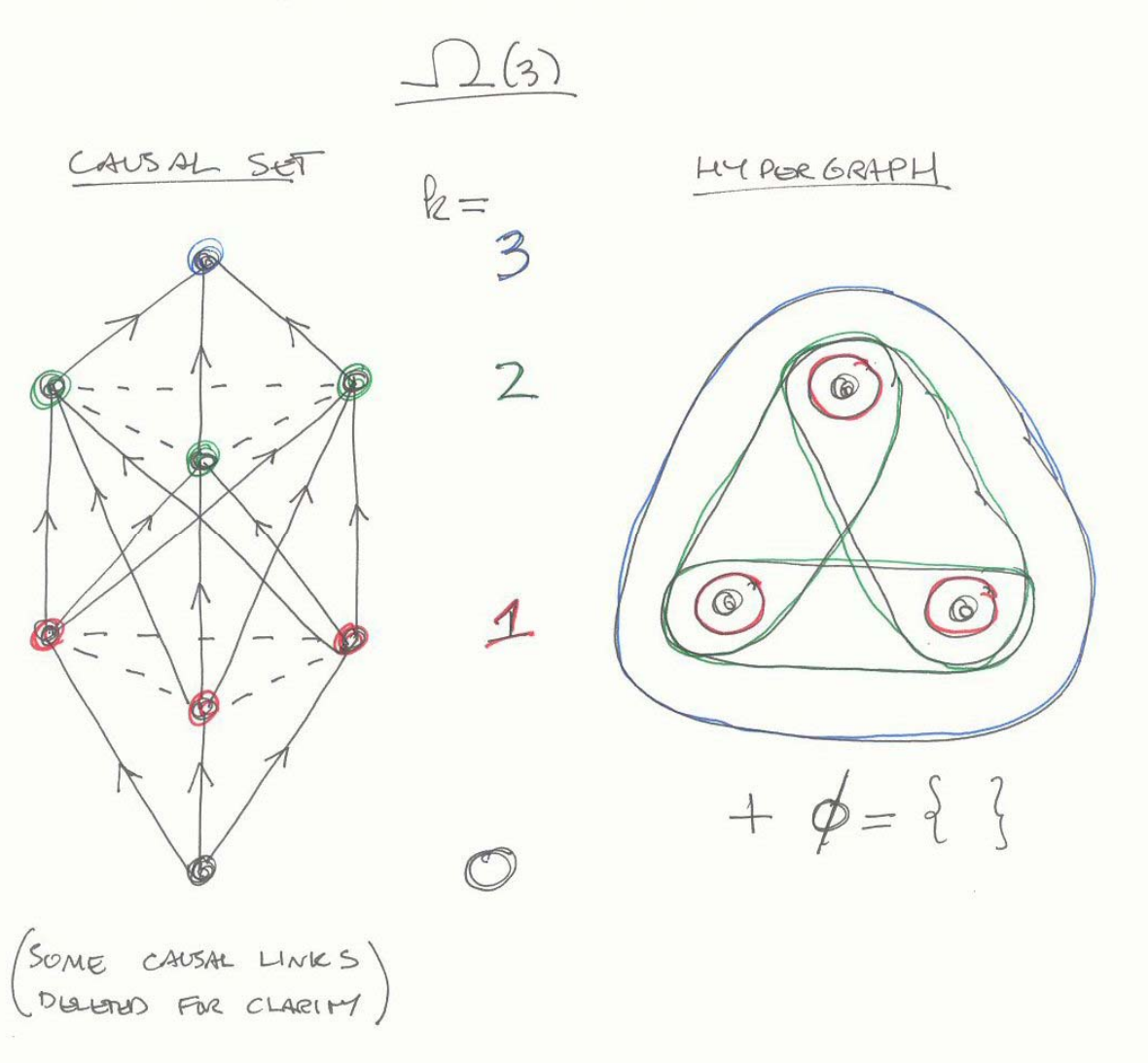
Three Results for the Omega Set

- ## The Cosmos is a Pancake

    There is a Big Bang and a Big Crunch. But maximum expansion is reached at a central bulge, which contains almost every point in the causal set. This is a very thin space-like sheet, 5×10^{17}s (15 billion light years) in diameter and about 2×10^{-13}s thick.

    This interval neatly splits fundamental interactions: strong and electromagnetic forces act over shorter time scales ($T_{QCD} < 10^{-22}$s, $T_{QED} \sim 10^{-14}$s–10^{-20}s), while weak forces take longer to act ($T_{weak} > 10^{-13}$s). This may be related to electro-weak symmetry breaking.

- ## The Local Structure is quasi-Galilean

    This is true not just for the Omega set, but for any causal set generated in this way. Relative to any point in the causal set there are points in its future and/or past light cones and space-like points. However there are no points in the interior of either the past or future light cone. This suggests an external parameter should be used to represent normal time evolution, rather than the causal direction internal to the cosmology.

- ## The Hubble Law is Observed

    (Almost) All points are in the central bulge, hence this is the generic residence of an astronomical observer. From this point of view, the Hubble Law of recession velocities holds, either exactly or approximately. Deviations from the Hubble Law can be controlled by the choice of a specific causal set chosen within the Omega Set "envelope."

Time index: global shape and central bulge

We denote by the same symbol an edge and the number of nodes in that edge. So A is the number of nodes in the edge A. We define the time index of the causal set point **α** as the number A, an integer: $k(α)=A$. This provides a well-ordering of the causal set points. Thus the causal set is composed of a number of space slices, labeled by $k=0,1,2 … N$.

To see the shape of the causal set, consider the number of points in any given slice. For the k_{th} slice, this is the binomial coefficient (N, k). For a large number N, this means that almost all the points are concentrated in a region of time index thickness ~sqrt(N). So there is a <u>central bulge</u> where almost all points can be found, with $k=N/2$; the causal set is "sparse" elsewhere.

The required value of N can be estimated as follows. Assume adjacent time slices are separated by t_P, the Planck time. Since almost all points are found in the central bulge, assume that corresponds to the present age of the universe, t_{now}. It follows that $N/2 = t_{now}/t_P$. The thickness of the central bulge is therefore approximately $t_{thick}=sqrt(N) t_P$:
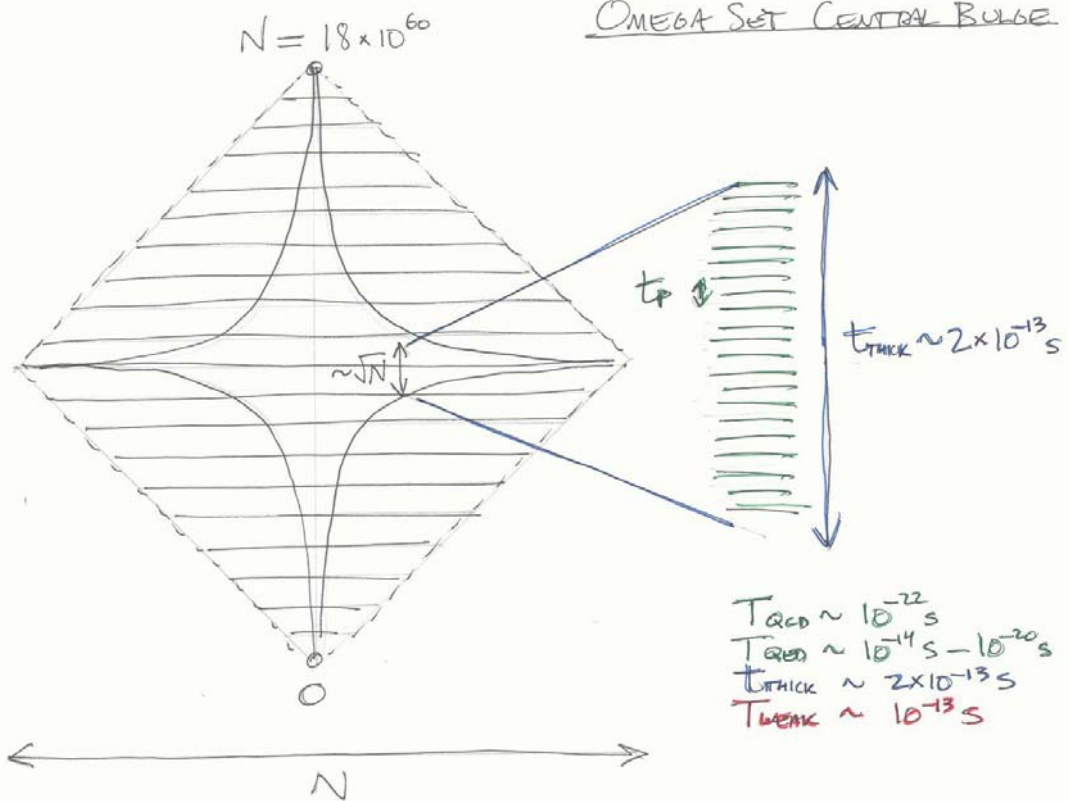
$$t_P = 5.3 \times 10^{-44} s$$
$$t_{now} = 15 \text{ billion years} = 4.7 \times 10^{17} s$$
$$N = 18 \times 10^{60}, \quad sqrt(N) = 4.2 \times 10^{30}$$
$$\mathbf{t_{thick} = sqrt(N)\, t_P = sqrt(2 t_{now} t_P) = 2.2 \times 10^{-13} s.}$$

Compare this to the fundamental interaction time scales:

$$\mathbf{T_{QCD} \sim 10^{-22} s < t_{thick},}$$
$$\mathbf{T_{QED} \sim (10^{-14} s – 10^{-20} s) < t_{thick},}$$
$$\mathbf{but \quad T_{weak} \sim 10^{-13} s \sim t_{thick}.}$$

This may be related to electro-weak symmetry breaking.

OMEGA SET CENTRAL BULGE

$N = 18 \times 10^{60}$

$\sim \sqrt{N}$

t_P

$t_{THICK} \sim 2 \times 10^{-13} s$

O

N

$T_{QCD} \sim 10^{-22} s$
$T_{QED} \sim 10^{-14} s - 10^{-20} s$
$t_{THICK} \sim 2 \times 10^{-13} s$
$T_{WEAK} \sim 10^{-13} s$

Distance measure: triangle inequality and local shape

For any two edges A and B we define a distance by

$$d(AB) = U(AB)-I(AB) = A+B-I,$$

where $U(AB)=A \cup B$ is their union and $I(AB)=A \cap B$ is their intersection. For two identical edges this would be zero; for two non-overlapping edges this is A+B. *(See elsewhere for a proof of the <u>Triangle Inequality</u>.)*

This provides a diameter for the causal set, for any time index k: $D(k)=2k$ if $k \leq N/2$ but $D(k)=N-2k$ if $k \geq N/2$. Thus the universe begins at k=0 with zero diameter, expands up to the present time and achieves maximal diameter N for k=N/2, and then contracts until it reaches zero diameter for k=N.

Now consider the analogue to the relativistic invariant $s^2=\Delta t^2-\Delta x^2$. For edges A and B, use $\Delta t=B-A$ and $\Delta x=A+B-I$, where $A \leq B$:

$s^2 = \Delta t^2-\Delta x^2 = (A-B)^2-(A+B-I)^2$
   $= 2I(A+B)-4AB \leq 2A(A+B)-4AB$, using $I \leq A$
   $\leq 2A(B+B)-4AB = 0$, so
$s^2 \leq 0$.

We get $s^2=0$ by choosing $A \subseteq B$, whence $\Delta x=B-A$ and $\Delta t=B-A$. So the subsets of any edge B are in the past light cone of the point **β.** There are no points in the future or past light cone interiors of **β**, but there are usually points in its present. The <u>Big Bang</u> (k=0) has all other points in its future light cone. The <u>Big Crunch</u> (k=N) has all other points in its past light cone. All other points have some points in their light cones and some points in their present. This may suggest that the causal direction internal to the cosmology is "suspect" as a conventional time evolution parameter.

$$\boxed{\text{HYPERGRAPH MEASURES}} \quad N-\text{NODE GRAPH}$$
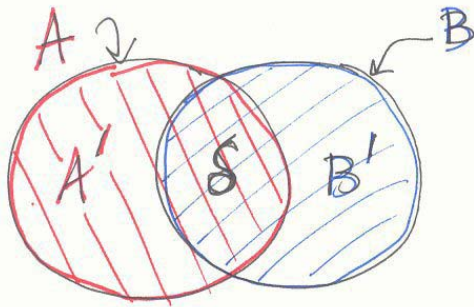
$$B' + \delta = B$$
$$A' + \delta = A \leq B$$
$$\delta = A \cap B \;,\; 0 \leq A, B \leq N$$
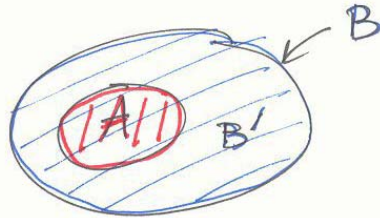
- <u>TIME</u> : $t(A) = A$, $t(B) = B$

$$\Delta t = B - A \;\; \in [0, N]$$

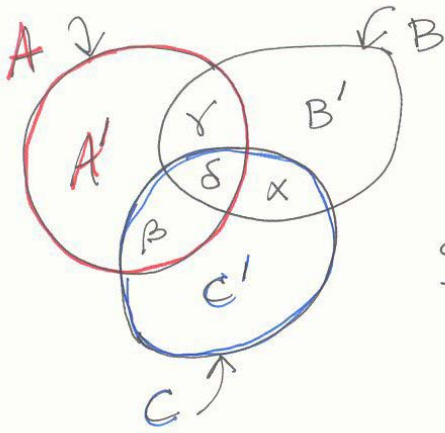- <u>DISTANCE</u> : $d(AB) = U - I = A' + B'$

$$\in [0, N]$$

※ <u>LIGHT CONE</u> :

$$\Delta t = d = B - A = B'$$
$$\Delta t^2 - d^2 = 0 \;.$$
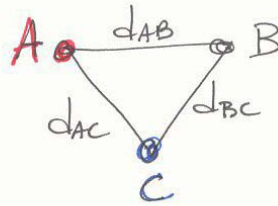
※ <u>IN GENERAL</u>, $\Delta t^2 \leq d^2 \;.$

LIGHT CONE INTERIORS ARE <u>EMPTY</u> .

$\boxed{\text{TRIANGLE INEQUALITY}}$

$A = A' + \beta + \gamma + \delta$
$B = B' + \alpha + \gamma + \delta$
$C = C' + \alpha + \beta + \delta$
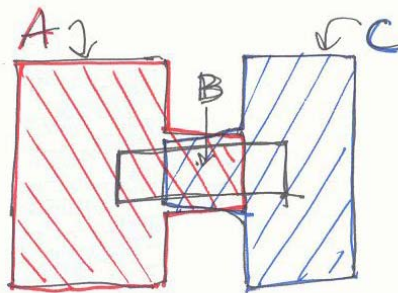
$$\underline{d_{AC} \leq d_{AB} + d_{BC}} \quad ?$$

○ $\underline{\text{PROOF}}$:

$d_{AB} = A' + \beta + B' + \alpha$
$d_{BC} = B' + \gamma + C' + \beta$
$d_{AC} = A' + \gamma + C' + \alpha$

$\left.\begin{array}{}\end{array}\right\}$ FROM $\underline{\text{DEFINITION}}$ OF $\underline{\text{DISTANCE}}$.

$d_{AB} + d_{BC} - d_{AC} = A' + \beta + B' + \alpha$
$+ B' + \gamma + C' + \beta$
$- A' - \gamma - C' - \alpha$

$$= 2(B' + \beta) \geq 0 \ . \quad \checkmark$$

○ $\underline{d_{AC} = d_{AB} + d_{BC}}$ $\underline{\text{IFF}}$ $B' = \beta = 0$

The Hubble Law

To derive the Hubble Law, consider two edges A and B such that **α** is in the past light cone of **β**, or A ⊆ B: A is the source and B is the observer. We note that U=B, I=A and d=B-A. The (current) age of the universe is B; this is also the maximum distance one can observe. The fractional distance of the source is therefore $\rho=(B-A)/B$.

One approach is to make an <u>ad-hoc definition</u> of the recession velocity:

$$v(AB) = 1-I(AB)/U(AB) = (U-I)/U \in \{0,1\}.$$

Since A ⊆ B, we get $v = (B-A)/B = \rho$. If we follow Hubble and say that v=Hd, we can see that H=1/B. This is constant for a given observer B, and equal to the inverse of the (current) age of the universe.

Alternately, <u>compare the scale factors of the universe</u>, R_B at time B and R_A at time A. Their ratio represents the dilation factor for light of wavelength λ traveling from A to B: $\lambda_B/\lambda_A=R_B/R_A$. Using the (incorrect!) classical Doppler shift formula with moving source and stationary observer we again get we get $v=\rho$. If we use the (correct!) relativistic formula we get something similar:

$$v=\rho(2-\rho)/[2-\rho(2-\rho)].$$

This agrees with the classical result for small distances, but is larger for large distances; it runs counter to the accelerating universe data. *(See elsewhere for a <u>graph of v vs. ρ</u>)…* But deviations from the Hubble Law can be controlled by the choice of a specific causal set chosen within the Omega Set "envelope."

Outlook for Future Work

Beyond the Omega Set

The OS is a complete hypergraph. Other possible causal sets are reached by deleting edges, but how should this be done? If the graph itself is not a complete graph, the nodes may be grouped into well-connected clusters (cliques). These correspond to points on the causal set, which is a subset of the OS and represents a different cosmology.

Quantum Field on the Graph

Consider a binary field on the graph nodes. Every edge corresponds to a classical state, a basis vector in the quantum state space, and a subspace spanned by subsets of that edge. A unique Hilbert space can thus be assigned to each point in the causal set, and this "covers" all the Hilbert spaces in the past light cone in a natural way; thus quantum information can flow in the causal set.

The Role of Quantum Observers

The links in the graph are abstract but always denote relations between the nodes. A link may be viewed as a distance between two space elements or as a relation between two quantum observers, which are represented by vectors in a generic Hilbert space. Generalizing the simple graph slightly, each link is the inner product of two such vectors; this allows an alternate, complementary description of the universe using quantum states rather than spacetime.